Beyond the mean field approximation for spin glasses

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We study the d-dimensional random Ising model using a suitable type of Bethe-Peierls approximation in the framework of the replica method. We take into account the correct interaction only inside replicated clusters of spins. Our ansatz is that the interaction of the borders of the clusters with the external world can be described via an effective interaction among replicas. The Bethe-Peierls model can be mapped into a single Ising model with a random Gaussian field, whose strength (related to the effective coupling between two replicas) is determined via a self-consistency equation. This allows us to obtain analytic estimates of the internal energy and of the critical temperature in d dimensions. [S1063-651X(96)00111-0]

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INTRODUCTION

The mean field solution and its improvements, such as the Bethe-Peierls approximation [1,2], give good approximations of the critical temperature and of the internal energy in many statistical models. We show that the same methods can be applied to spin glasses by considering the overlap among replicas instead of the magnetization. Indeed, the appropriate ansatz for spin glasses is assuming that the effect of the thermal bath on a replicated cluster of neighbors' spins produces an effective coupling among replicas. We shall give an *a posteriori* justification of such a hypothesis by proving that the Sherrington-Kirkpatrick (SK) model [3] is recovered in the limit of infinite dimension.

This paper considers the Ising model with independent random nearest-neighbor coupling J_{ij} in the absence of external magnetic field. Our main result is that, in the Bethe-Peierls approximation, this model is equivalent to a single Ising model with a random Gaussian field whose strength is related to the effective coupling between two replicas. We thus obtain an estimate of the internal energy and of the critical temperature in any dimension.

In Sec. I we introduce the Bethe-Peierls ansatz for spin glasses in the framework of the replica method. In Sec. II we prove that this ansatz leads to the SK model when $d \to \infty$. In Sec. III we show that, under the hypothesis of no replica symmetry breaking, the d-dimensional model can be mapped into a single Ising model with random Gaussian field. We also explicitly compute the replica symmetry solution for the internal energy in d dimensions for the spin glass with dichotomic random coupling $J = \pm 1$. In Sec. IV we show that our method allows us to compute in a simple way the critical temperature $T_c(d)$. The result is very accurate at high dimension. In Sec. V we discuss the possibility of using our ideas to implement a clever numerical scheme for determining internal energy and critical temperature of d-dimensional spin glasses.

I. BETHE-PEIERLS ANSATZ FOR SPIN GLASSES

The partition function of the Ising models on a lattice of N sites with nearest-neighbor couplings J_{ij} which are independent identically distributed random variables, in the absence of external magnetic field, is

$$Z_N(\beta,\{J_{ij}\}) = \sum_{\{s\}} \prod_{(i,j)} \exp(\beta J_{ij}\sigma_i\sigma_j), \tag{1.1}$$

where the sum runs over the 2^N spin configurations $\{s\}$, and the product over the dN nearest-neighbor sites (i,j).

In the thermodynamic limit almost all disorder realizations have the same free energy, i.e., the quenched free energy

$$f = -\lim_{N\to\infty} \frac{1}{\beta N} \overline{\ln Z_N}, \tag{1.2}$$

where $\overline{A}$ indicates the average of an observable A over the distribution of the random coupling $P(J_{\underline{i,j}})$. In the following we assume that the $P(J_{ij})$ is such that $\overline{J_{ij}} = 0$ and $\overline{J_{ij}^2} = 1$.

On the other hand, it is trivial to compute the so-called annealed free energy

$$f_a = -\lim_{N\to\infty} \frac{1}{\beta N} \ln \overline{Z_N}, \tag{1.3}$$

corresponding to the free energy of a system where the random coupling are not quenched but can thermalize with a relaxation time comparable to that of the spin variables. For instance, in the case of dichotomic random coupling $J_{ij} = \pm 1$ with equal probability, one has

$$f_a = -\beta^{-1}(\ln 2 + d \ln\cosh\beta), \tag{1.4a}$$

while for Gaussian coupling, i.e., $P(J_{ij}) = (2\pi)^{-1/2}\exp(J_{ij}^2/2)$, one has

$$f_a = -\beta^{-1}\left(\ln 2 + \frac{\beta^2 d}{2}\right). \tag{1.4b}$$

However, f_a is in general very different from the quenched free energy. In order to compute (1.1), it is convenient to use the replica trick [4]. Let us thus consider n noninteracting replicas of the disordered system labeled by $\alpha=1,...,n$. The corresponding partition function is

$$Z^n=\sum_{\{\mathbf{s}\}} \exp\left(\beta\sum_{\alpha=1}^{n} \sum_{(i,j)} J_{ij}\sigma_i^{(\alpha)}\sigma_j^{(\alpha)}\right), \tag{1.5}$$

where the sum runs over the 2^{Nn} spin configurations $\{\mathbf{s}\}$ of the replicas,

$$\{\mathbf{s}\}\equiv\{s^{(1)}\},\ldots,\{s^{(n)}\},$$

with

$$s^{(\alpha)}=(\sigma_1^{(\alpha)},\sigma_2^{(\alpha)},\ldots,\sigma_N^{(\alpha)}).$$

After having performed the average $\overline{Z^n}$ and found an analytic continuation at real n values, the quenched free energy is given by

$$\overline{\ln Z}=\lim_{n\to 0}\frac{1}{n}\ln\overline{Z^n}. \tag{1.6}$$

Even in two dimensions, there is no exact solution for this problem. The first nontrivial approximations of the quenched free energy can be obtained either by constrained annealed average [5] or by improved mean field approximations of the Bethe-Peierls type. Recently we have introduced such an approximation in the dual lattice made of square plaquettes in two dimensions [6]. However, there is no solution of the self-consistency equation at low temperature and it is not trivial to generalize the approach at higher dimensions. For systems with diluted quenched disorder, a different type of improved Bethe-Peierls approximation (the cluster variation method) has been studied in [7] without using the replica approach.

In this paper we want to work directly on the real lattice, by taking into account the correct interactions inside a pile of replicated clusters made of a central spin σ_0 and of its $2d$ nearest neighbor $\{\sigma_k\}$, and by considering only an effective interaction with the external world. Note that in the $2d$ case, the clusters are crosses made of five spins.

Separating the two contributions (crosses plus external world) in the partition function, we get

$$\overline{Z^n}=\sum_{\{\mathbf{s}_{\rm cr}\}}\left[\overline{\exp\left(\beta\sum_{\alpha=1}^{n}\sum_{k=1}^{2d} J_k\sigma_0^{(\alpha)}\sigma_k^{(\alpha)}\right)}\right.$$
$$\left.\times\sum_{\{\mathbf{s}_{\rm ext}\}}\overline{\exp\left(\beta\sum_{\alpha=1}^{n}\sum_{(i,j)\neq(0,k)} J_{ij}\sigma_i^{(\alpha)}\sigma_j^{(\alpha)}\right)}\right], \tag{1.7}$$

where

$$J_k\equiv J_{0k}$$

are the coupling between the central spin of the cross and its neighbors on the border. The first sum in (1.7) runs over the $2^{(2d+1)n}$ spin configurations $\{\mathbf{s}_{\rm cr}\}$ of the replicated crosses labeled by $(\sigma_0^\alpha,\sigma_1^\alpha,\ldots,\sigma_{2d}^\alpha)$ with $\alpha=1,\ldots,n$ while the second sum runs over all the other spins. The expression obtained by computing the second sum depends only on the $2dn$ lateral spins σ_k. The correct Bethe-Peierls ansatz is given by the assumption that the interaction among the lateral spins of the replicated crosses and the external world forces an effective interaction among different replicas with a constant $\mu_{\alpha\beta}$ that should be determined via a self-consistency equation. In other terms, our Bethe-Peierls ansatz is

$$\sum_{\{\mathbf{s}_{\rm ext}\}}\overline{\exp\left(\beta\sum_{\alpha=1}^{n}\sum_{(i,j\neq 0,k)} J_{ij}\sigma_i^{(\alpha)}\sigma_j^{(\alpha)}\right)}$$
$$=K(\beta)\exp\left(\sum_{\alpha>\beta}\mu_{\alpha\beta}\sum_{k=1}^{2d}\sigma_k^{(\alpha)}\sigma_k^{(\beta)}\right), \tag{1.8}$$

where $K(\beta)$ is a multiplicative constant which depends on the temperature but not on the lateral spins. One expects that $\mu_{\alpha\beta}=0$ in the high-temperature phase, while it must have a nonzero value in the glassy phase.

Therefore, instead of (1.7), we have to compute an effective partition function Z_n,

$$Z_n=\sum_{\{\mathbf{s}_{\rm cr}\}}\prod_{k=1}^{2d}\overline{\exp\left(\beta\sum_{\alpha=1}^{n} J_k\sigma_0^{(\alpha)}\sigma_k^{(\alpha)}\right)}$$
$$\times\exp\left(\sum_{\alpha>\beta}\mu_{\alpha\beta}\sigma_k^{(\alpha)}\sigma_k^{(\beta)}\right). \tag{1.9}$$

A further simplification can be reached for dichotomic coupling $J_{ij}=\pm 1$ where one can perform the gauge transformation $\sigma_k^{(\alpha)}\to J_k\sigma_k^{(\alpha)}$ on the lateral spins, leaving the free energy unchanged. In this case the averaged partition function (1.9) becomes

$$Z_n=\sum_{\{\mathbf{s}_{\rm cr}\}}\prod_{k=1}^{2d}\exp\left(\beta\sum_{\alpha=1}^{n}\sigma_0^{(\alpha)}\sigma_k^{(\alpha)}\right)\exp\left(\sum_{\alpha>\beta}\mu_{\alpha\beta}\sigma_k^{(\alpha)}\sigma_k^{(\beta)}\right). \tag{1.10}$$

This relation implies the rather surprising result that a nondisordered Ising system exhibits the same behavior of a spin glass if one imposes the appropriate interaction among different replicas.

At this point, the effective coupling $\mu^*_{\alpha\beta}(\beta)$ is given by the self-consistency equation

$$\lim_{n\to 0}\langle\sigma_k^{(\alpha)}\sigma_k^{(\beta)}\rangle_n=\lim_{n\to 0}\langle\sigma_0^{(\alpha)}\sigma_0^{(\beta)}\rangle_n, \tag{1.11}$$

where $\langle\,\rangle_n$ represents the thermal average over the replicated system. Then, the Bethe-Peierls estimate of the internal energy is

$$U_{\rm BP}(\beta)=\lim_{n\to 0}-\frac{1}{2n}\left[\frac{\partial}{\partial\beta}\ln Z_n(\mu_{\alpha\beta},\beta)\right]_{\mu_{\alpha\beta}=\mu^*_{\alpha\beta}}. \tag{1.12}$$

Let us anticipate that the Bethe-Peierls approximation predicts a phase transition at a critical temperature $T_c(d)$ above which $\mu^*_{\alpha\beta}=0$. As a consequence the Bethe-Peierls solution coincides with the annealed one in the high-temperature phase, i.e.,

$$U_{\rm BP}=\frac{d}{d\beta}\left[\beta f_a(\beta)\right]\quad\text{for } \beta<\beta_c\,. \tag{1.13}$$

II. THE INFINITE-DIMENSIONAL LIMIT

The model defined by (1.9) becomes the infinite-range SK model [3] in the limit $d\to\infty$. This result has great importance since it provides good evidence that we have chosen the correct Bethe-Peierls ansatz for spin glasses. In this section we prove that the self-consistency equation (1.11) in the limit $d\to\infty$ gives the equation for the overlap of the SK model.

Let us recall that the averaged partition function of the infinite-range SK model after some simple algebraic manipulation becomes

$$\overline{Z}=(\overline{Z})^n\left[\max_{q_{\alpha\beta}}\frac{1}{2^n}\sum_{\{\sigma\}}\exp\beta^2\left(\sum_{\alpha>\beta}q_{\alpha\beta}\sigma^{(\alpha)}\sigma^{(\beta)}-\frac{q^2_{\alpha\beta}}{2}\right)\right]^N, \tag{2.1}$$

where the sum is on the 2^n realizations $\{\sigma\}$ of the n spins $\sigma^{(1)},\ldots,\sigma^{(n)}$. In the high-temperature phase $T\geqslant T_c$, one has $q_{\alpha\beta}=0$ so that $\overline{Z^n}=(\overline{Z})^n$, while in the glassy phase one has a nontrivial overlap $q_{\alpha\beta}=q^*_{\alpha\beta}(T)$ which maximizes $(\overline{Z})^n$, that is,

$$q_{\alpha\beta}=\langle\sigma^\alpha\sigma^\beta\rangle\equiv\frac{\Sigma_{\{\sigma\}}\sigma^{(\alpha)}\sigma^{(\beta)}\exp(\beta^2\Sigma_{\alpha>\beta}q_{\alpha\beta}\sigma^{(\alpha)}\sigma^{(\beta)})}{\Sigma_{\{\sigma\}}\exp(\beta^2\Sigma_{\alpha>\beta}q_{\alpha\beta}\sigma^{(\alpha)}\sigma^{(\beta)})}. \tag{2.2}$$

In order to get the correct $d\to\infty$ limit of the self-consistency equations (1.11), we should use the rescaling

$$\beta\to\frac{\beta}{\sqrt{2d}},\quad \mu_{\alpha\beta}\to\beta^2\mu_{\alpha\beta}\,. \tag{2.3}$$

Now, the disorder average in Z_n is easily performed since at large d the first exponential in (1.9) can be expanded in Taylor series up to the second order, so that

$$\begin{aligned}\overline{\exp[\beta(2d)]^{-1/2}J_kS_k}&=\overline{1+\beta(2d)^{-1/2}J_kS_k+\frac{\beta^2}{4d}J_k^2S_k^2}+O(d^{-3/2})\\&=1+\frac{\beta^2}{4d}S_k^2+O(d^{-3/2})=\exp\left(\frac{\beta^2}{4d}S_k^2\right)+O(d^{-3/2}),\end{aligned}$$

where $S_k\equiv\Sigma_\alpha\sigma_0^{(\alpha)}\sigma_k^{(\alpha)}$. The distribution of the coupling is irrelevant provided that $\overline{J}=0$ and $\overline{J^2}=1$. Therefore after a small correction $O(d^{-3/2})$, the partition function (1.9) becomes

$$Z_n=\sum_{\{\mathbf{s}_{\rm cr}\}}\exp\beta^2\left(\sum_{\alpha>\beta}\sigma_0^{(\alpha)}\sigma_0^{(\beta)}\frac{1}{2d}\sum_{k=1}^{2d}\sigma_k^{(\alpha)}\sigma_k^{(\beta)}+\mu_{\alpha\beta}\sum_{k=1}^{2d}\sigma_k^{(\alpha)}\sigma_k^{(\beta)}\right) \tag{2.4}$$

implying that

$$\langle\sigma_k^{(\alpha)}\sigma_k^{(\beta)}\rangle=\frac{\Sigma_{\{\sigma_k\}}\sigma_k^{(\alpha)}\sigma_k^{(\beta)}\exp(\beta^2\Sigma_{\alpha>\beta}\mu_{\alpha\beta}\sigma_k^{(\alpha)}\sigma_k^{(\beta)})}{\Sigma_{\{\sigma_k\}}\exp(\beta^2\Sigma_{\alpha>\beta}\mu_{\alpha\beta}\sigma_k^{(\alpha)}\sigma_k^{(\beta)})}+O(d^{-1/2}), \tag{2.5}$$

where k is one of the lateral sites of the d-dimensional crosses and the sum is on the 2^n realizations σ_k of the n spins $\sigma_k^{(1)},\ldots,\sigma_k^{(n)}$.

On the other hand, in the limit $d\to\infty$, the corresponding relation for the central spins can be written as

$$\langle\sigma_0^{(\alpha)}\sigma_0^{(\beta)}\rangle=\frac{\Sigma_{\{\sigma_0\}}\sigma_0^{(\alpha)}\sigma_0^{(\beta)}\exp(\beta^2\Sigma_{\alpha>\beta}\langle\sigma_k^{(\alpha)}\sigma_k^{(\beta)}\rangle\sigma_0^{(\alpha)}\sigma_0^{(\beta)})}{\Sigma_{\{\sigma_0\}}\exp(\beta^2\Sigma_{\alpha>\beta}\langle\sigma_k^{(\alpha)}\sigma_k^{(\beta)}\rangle\sigma_0^{(\alpha)}\sigma_0^{(\beta)})} \tag{2.6}$$

since one has

$$\langle\sigma_k^{(\alpha)}\sigma_k^{(\beta)}\rangle=\lim_{d\to\infty}\frac{1}{2d}\sum_{k=1}^{2d}\sigma_k^{(\alpha)}\sigma_k^{(\beta)}\,. \tag{2.7}$$

A direct comparison of (2.6) and (2.5) shows that the self-consistency equation (1.11) is satisfied only if

$$\mu_{\alpha\beta}=\langle\sigma_k^{(\alpha)}\sigma_k^{(\beta)}\rangle$$

that is the equation for the overlap of the SK model. We can thus identify the coupling $\mu_{\alpha\beta}$ with the overlap $q_{\alpha\beta}$ for $d\to\infty$.

III. REPLICA SYMMETRY SOLUTION IN THE BETHE-PEIERLS APPROXIMATION

It is possible to obtain the replica symmetry solution of a d-dimensional spin glass in the Bethe-Peierls approximation. We must note that in the case $\mu_{\alpha\beta}=\mu$, the averaged partition function (1.9) of n replicated crosses is

$$Z_n=\sum_{\{\mathbf{s}_{\rm cr}\}}\prod_{k=1}^{2d}\overline{\exp\left(\beta J_k\sum_{\alpha=1}^{n}\sigma_0^{(\alpha)}\sigma_k^{(\alpha)}\right)}\exp\left(\mu\sum_{\alpha>\beta}\sigma_k^{(\alpha)}\sigma_k^{(\beta)}\right). \tag{3.1}$$

Except for constant multiplicative factors, it can also be written as

$$Z_n=\sum_{\{\mathbf{s}_{\rm cr}\}}\prod_{k=1}^{2d}\overline{\exp\left(\beta J_k\sum_{\alpha=1}^{n}\sigma_0^{(\alpha)}\sigma_k^{(\alpha)}\right)}\exp\frac{\mu}{2}\left(\sum_{\alpha}\sigma_k^{(\alpha)}\right)^2 \tag{3.2}$$

that is bilinear in σ_k. In order to linearize (3.2), we should use the standard Gaussian identity

$$\exp(x^2/2)=\frac{1}{(2\pi)^{1/2}}\int_{\infty}^{\infty}d\omega\,\exp(-\omega^2/2)\exp(\omega x)$$

so that one has

$$Z_n = \sum_{\{\mathbf{s}_{\rm cr}\}} \prod_{k=1}^{2d} \overline{\exp\left(\sum_{\alpha=1}^{n} (\beta J_k \sigma_0^{(\alpha)} \sigma_k^{(\alpha)} + \mu^{1/2} \omega_k \sigma_k^{(\alpha)}) \right)}, \tag{3.3}$$

where

$$\overline{\psi} = \frac{1}{(2\pi)^{d/2}} \int \cdots \int \prod_{k=1}^{2d} d\omega_k e^{-\omega_k^2/2} \prod_{k=1}^{2d} P(J_k) dJ_k \psi \tag{3.4}$$

indicates now the average over the standard Gaussian variables ω_k and over the coupling J_k between central and lateral spins.

This transformation has the advantage of allowing for a factorization of the product over the replicas in (3.3), implying that

$$Z_n = \overline{\Phi^n}, \tag{3.5}$$

with

$$\Phi = \sum_{\{\mathbf{s}_{\rm cr}\}} \prod_{k=1}^{2d} \exp(\beta J_k \sigma_0 \sigma_k + \mu^{1/2} \omega_k s_k). \tag{3.6}$$

This is the main result of the section. It establishes that, in the Bethe-Peierls approximation, the replica symmetry solution is equivalent to that of a single Ising model with a random Gaussian field applied to the boundaries of the cross. This field has a strength related to the coupling among replicas and describes the interaction of the cluster of $2d+1$ spins with the external world.

The explicit sum over the lateral spins σ_k gives

$$Z_n = \overline{\left(\sum_{\{s_{\rm cr}\}} W_\mu(\sigma_0) \right)^n}, \tag{3.7}$$

where W_μ is the non-normalized weight of the central spin,

$$W_\mu(\sigma_0) = \prod_{k=1}^{2d} 2\cosh(\beta J_k \sigma_0 + \mu^{1/2} \omega_k), \tag{3.8}$$

obtained after summing over the configurations of the $2d$ lateral spins σ_k. The probability of the central spin thus is

$$P_\mu(\sigma_0) = \frac{W_\mu(\sigma_0)}{W_\mu(\sigma_0=1) + W_\mu(\sigma_0=-1)} \tag{3.8$'$}$$

and is itself a random quantity depending on the $2d$ random Gaussian fields and the $2d$ random coupling J_k.

Because of the replica symmetry, the self-consistency equation (1.11) for determining μ^*, and so the needed strength of the random field, assumes the simpler form

$$\overline{\langle \sigma_0 \rangle^2} = \overline{\langle \sigma_1 \rangle^2}, \tag{3.9}$$

where the thermal average of the central spin is

$$\langle \sigma_0 \rangle = \sum_{\sigma_0 = \pm 1} \sigma_0 P_\mu(\sigma_0) \tag{3.10}$$

and the thermal average of one of the lateral spins is

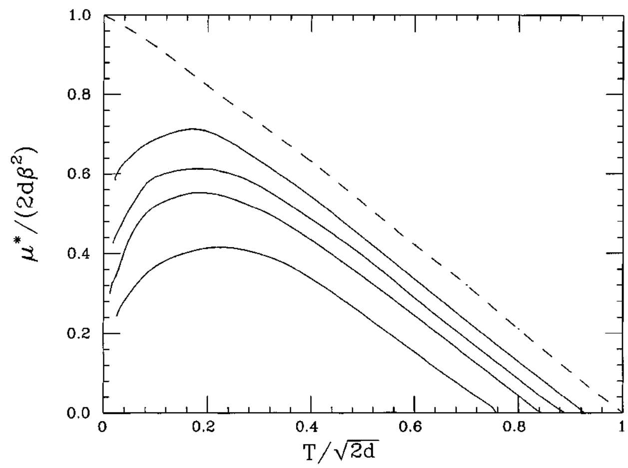

FIG. 1. Replica symmetry solution of the self-consistency equation $\mu^*/(2d\beta^2)$ as a function of the rescaled temperature $T/(2d)^{1/2}$ for the $\pm J$ model at $d=2,3,4,6$. The larger the dimension, the higher the corresponding line. The dashed line indicates the infinite-dimensional limit (overlap of the SK model).

$$\langle \sigma_1 \rangle = \sum_{\sigma_0 = \pm 1} \tanh(\beta J_k \sigma_0 + \mu^{1/2} \omega_1) P_\mu(\sigma_0). \tag{3.11}$$

In order to find the internal energy we have to compute

$$\lim_{n \to 0} \frac{1}{n} \ln Z_n = \overline{\ln \sum_{\sigma_0 = \pm 1} \prod_{k=1}^{2d} 2\cosh(\beta J_k \sigma_0 + \mu^{1/2} \omega_k)} \tag{3.12}$$

and then, following (1.12), the internal energy is given by a derivative at $\mu = \mu^*$, the solution of (3.9),

$$U_{\rm BP}(\beta) = -d \overline{\sum_{\sigma_0 = \pm 1} J_k \sigma_0 \tanh(\beta J_k \sigma_0 + \mu^{1/2} \omega_1) P_\mu(\sigma_0)}. \tag{3.13}$$

It is worth stressing that in the limit $d \to \infty$, the self-consistency equation (3.9) becomes, by virtue of the results of Sec. II,

$$\mu = \lim_{d \to \infty} \overline{\langle \sigma_1 \rangle^2}. \tag{3.14}$$

The above expression, after performing the rescaling (2.3), gives the replica symmetry solution for the overlap of the SK model in the glassy phase,

$$\mu = \overline{\tanh^2(\beta \omega \mu^{1/2})}, \tag{3.15}$$

where ω is again a standard Gaussian.

For the $\pm J$ model, after the gauge transformation $J_k \sigma_k \to \sigma_k$, the probability $P_\mu(\sigma_0)$ depends only on the Gaussian fields and is independent of the coupling J_k. We can thus set $J_k = 1$ in the formulas from (3.3) to (3.13) and the average (3.3) should be taken only over the $2d$ Gaussian variables ω_k.

Figure 1 shows the replica symmetry solution $2dT^2\mu^*(T)$ as a function of the rescaled temperature $(2d)^{1/2}T$ at $d=2,3,4,6$ for the $\pm J$ model. The effective rep-

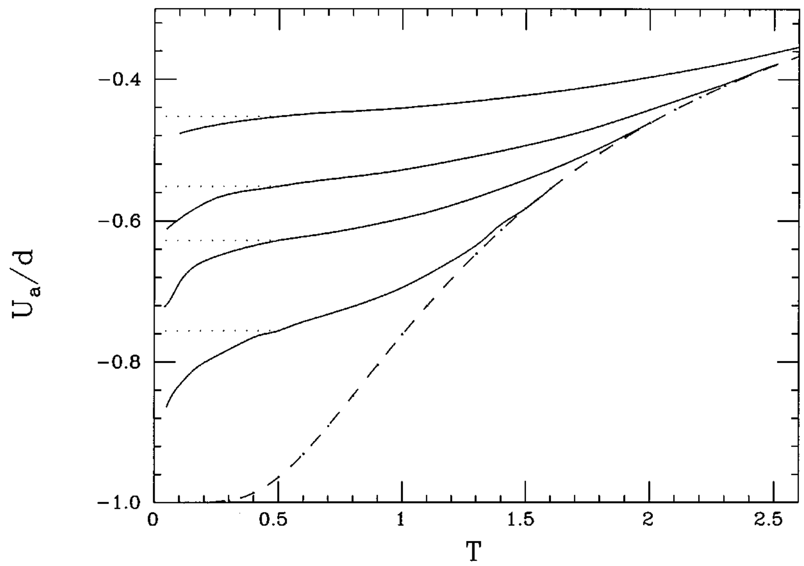

FIG. 2. Annealed internal energy $U_a/d = \tanh(\beta)$ (dashed line) and the Bethe-Peierls solutions $U_{\rm BP}/d$ (full lines) versus temperature $T=\beta^{-1}$ for the $\pm J$ model at $d=2,3,4,6$. The larger the dimension, the higher the corresponding line. The dotted lines are the estimates of the ground state energy obtained by imposing that μ^*/β^2 is a nondecreasing function of the temperature.

lica coupling μ^* vanishes above the critical temperature $T_c(d)$ as we shall discuss in the next section. As a consequence the internal energy $U_{\rm BP}$ is equal to the annealed internal energy at $T \geq T_c$. It is interesting to note that below $T' \approx 0.5$, $\mu^*(T)T^2$ decreases.

The reason can be understood by a simple qualitative argument. Consider the model defined by (3.12) with $J_k=1$ which is originated by the $\pm J$ model. On the lateral spins, there is a competition between the random field $\mu^{1/2}\omega_k\sigma_k$ and the ferromagnetic interaction $\beta\sigma_0\sigma_k$. This gives origin to a frustration of the system below T_c. However, if the temperature is very low, the ferromagnetic interaction dominates and we expect that the work μ^*/β^2 necessary to win the tendency of the spin σ_k to align with the field vanishes. In correspondence the system would become ferromagnetic with a ground state $U_0=-d$. Such a regime is clearly unphysical, and one can trust in our results only when the work μ^*/β^2 made to destroy the long-range order in the glassy phase is a nonincreasing function of the temperature, i.e., for $T \geq T'$.

The internal energy $U_{\rm BP}(T)/d$ is shown in Fig. 2 for the $\pm J$ model at $d=2,3,4,6$. An estimate of the ground state energy U_0 can be obtained by $U_{\rm BP}(T')$ as previously argued. Using this hypothesis we get

$$U_0=-1.51 \quad \text{at } d=2,$$

$$U_0=-1.88 \quad \text{at } d=3,$$

$$U_0=-2.204 \quad \text{at } d=4,$$

$$U_0=-2.718 \quad \text{at } d=6.$$

At $d=2$ we can compare our analytic estimate with the numerical result [8] $U_0=-1.404$.

It is an open issue to understand whether better estimates can be obtained via (1.12) with a replica symmetry breaking solution $\mu^*_{\alpha\beta}$.

IV. PHASE TRANSITION AND CRITICAL TEMPERATURE IN FINITE DIMENSION

The Bethe-Peierls method and its improvements are able to give accurate estimates the critical temperature of disordered systems. In a replica symmetry approach, good analytic results have been obtained for diluted spin glasses [9] and other randomly frustrated systems with finite connectivity [10].

In the framework of the results of the preceding section, we should note that at the transition point, the order parameter μ^* vanishes. Therefore the critical temperature can be computed from (3.3) considering only the first order of its expansion in μ^*.

Let us first compute the thermal average of the central spin

$$\langle \sigma_0 \rangle = \mu^{1/2} \tanh(\beta J_k) \sum_{k=1}^{2d} \omega_k + O(\mu). \tag{4.1}$$

Since this expression appears in (3.3) only in a squared form it is not necessary to compute higher orders than $\mu^{1/2}$. Analogously, the thermal average of one of the lateral spins is

$$\langle \sigma_1 \rangle = \mu^{1/2} \tanh(\beta J_1) \sum_{k=2}^{2d} \tanh^2(\beta J_k)\,\omega_k + \mu^{1/2}\omega_1 + O(\mu). \tag{4.2}$$

Inserting this expression in the consistency equation (3.9) one obtains

$$\mu 2d\,\overline{\tanh^2(\beta J)} = \mu(2d-1)\,\overline{\tanh^2(\beta J)}\,(\overline{\tanh^2(\beta J)})^2 + \mu, \tag{4.3}$$

where J is one of the couplings. Equation (4.3) gives the critical temperature $T_c=\beta_c^{-1}$ as a function of the dimension

$$\overline{\tanh^2(\beta_c J_k)} = \frac{1}{2d-1}. \tag{4.4}$$

In the case of the $\pm J$ model, this equation becomes

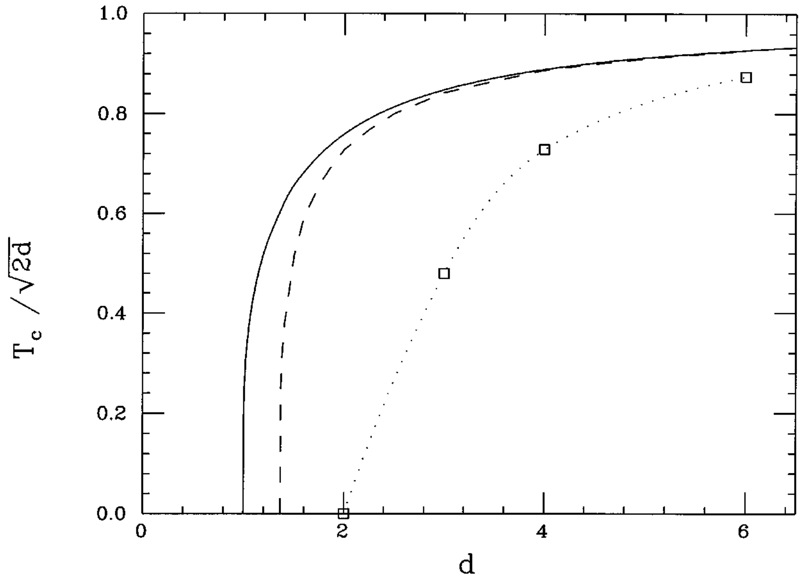

FIG. 3. Rescaled critical temperature $(2d)^{-1/2}T_c$ versus the dimension d. The Bethe-Peierls solution for the $\pm J$ model given by (4.5) is indicated by a full line. The improved estimate obtained by (4.6) where the cluster is a plaquette of four spins instead of a central spin is indicated by a dashed line. The squares are the numerical values of $(2d)^{-1/2}T_c$ for $d=2,3,4,6$ joined by a dotted line.

$$\tanh^2(\beta_c)=\frac{1}{2d-1}. \tag{4.5}$$

In Fig. 3 we compare the Bethe-Peierls critical temperature (4.5) with the numerical result obtained in the literature for the $\pm J$ model [11].

We have also computed the critical temperature of the Gaussian model via a numerical solution of (4.4). In this case

$$T_c=1.19 \quad \text{at } d=2,$$

$$T_c=1.81 \quad \text{at } d=3 \ \ (\text{numerical result } T_c=1.0),$$

$$T_c=2.28 \quad \text{at } d=4 \ \ (\text{numerical result } T_c=1.8),$$

$$T_c=2.67 \quad \text{at } d=5,$$

$$T_c=3.06 \quad \text{at } d=6.$$

Let us remark that T_c is finite in $d=2$. This spurious transition is a typical and well-known effect of mean field approximations. In fact, the Bethe-Peierls approximation gives a lower critical dimensionality $d_c=1$, where $T_c=0$, while there is good numerical evidence that $d_c=2$. On the other hand, the higher the dimensionality, the better our estimates. In the limit of infinite dimension, after the usual rescaling (2.3) of the temperature, from (4.5) one obtains $\beta_c=1$, which is the critical temperature of the SK model.

It is possible to improve the estimate of the critical temperature in a systematic way by considering a larger cluster instead of a cross made of a single central spin σ_0 and of its $2d$ neighbor σ_k, as we shall discuss in the conclusions.

For instance, we have considered a plaquette of four spins plus the $8(d-1)$ spins that are its nearest neighbors in the $\pm J$ model. Applying our Bethe-Peierls ansatz to the replicated plaquettes, the equation for the critical temperature $T_c(d)$ is again given by the solution of a rational function of $t\equiv\tanh(\beta)$. After a lengthy but trivial calculation one has

$$1+(2d-3)t^4-2(d-1)t^2+2(d-1)(t^4-t^2)t^2 \times\left[\frac{(1+t^2)^4+(1+t^4)^2}{(1+t^2)^2(1+t^4)^2}+\frac{2t^2}{(1+t^4)^2}\right]=0. \tag{4.6}$$

In this case, the lower critical dimension is $d_c=\frac{15}{11}$ instead of $d_c=1$ found for the cross. The critical temperature $T_c(d)$ obtained by (4.6) is shown in Fig. 3, too.

In our opinion, looking at increasingly larger clusters it is possible to determine the critical temperature of a d-dimensional spin glass as the zeros of rational functions of $t=\tanh(\beta)$ reaching an accuracy much larger than that given by direct numerical methods. Moreover, one can also hope to find a converging sequence of lower critical dimensions, simply considering the zeros of the rational functions with $t=1$ (i.e., $T_c=0$).

V. CONCLUSIONS AND PERSPECTIVES

The properties of finite-dimensional Ising spin glasses are largely unknown. The lower critical dimension itself is not known although most numerical simulations indicate $d=3$ as the lowest dimension which exhibits a glassy phase at finite temperature. Furthermore, even if the glassy phase is present, the existence of replica symmetry breaking at low dimensionality is still controversial. All that is a clear indication of the difficulties encountered when one tries to extract information directly from the model.

In our approach we simplify the task. Indeed, when we assume replica symmetry, our approximation reduces itself to the study of the quenched model

$$\Phi=\sum_{\{s_{\rm cr}\}}\prod_{k=1}^{2d}\exp(\beta J_k\sigma_0\sigma_k+\mu^{1/2}\omega_k\sigma_k), \tag{5.1}$$

where both the ω_k and the J_k are quenched variables. It should be noticed that in this model, one only deals with $2d+1$ spins and $4d$ quenched variables at most. The model is completed by the self-consistency equation (3.9) that we rewrite here as

$$\overline{\langle \sigma_0 \rangle^2} = \frac{\Sigma_k \overline{\langle \sigma_k \rangle^2}}{\Sigma_k 1}, \tag{5.2}$$

where the second term of (5.2) is the mean of the overlap on the lateral spin.

The validity of our approach stems from the possibility of a systematic improvement. Following a standard technique we can replace (5.1) by

$$Z = \sum_{\{s\}} \prod_{l,l'} \exp(\beta J_{ll'} \sigma_l \sigma_{l'}) \prod_{l,k} \exp(\beta J_{lk} \sigma_l \sigma_k) \times \prod_k \exp(\mu^{1/2} \omega_k \sigma_k), \tag{5.3}$$

where the first product is on all the first neighbor spins of a hypercube, the second product on the couples of spins formed by lateral spins labeled by k and their first neighbor on the faces of the hypercube, and the third product is simply on the lateral spins. This model is completed by the self-consistency equation

$$\frac{\Sigma_l \overline{\langle \sigma_l \rangle^2}}{\Sigma_l 1} = \frac{\Sigma_k \overline{\langle \sigma_k \rangle^2}}{\Sigma_k 1}, \tag{5.4}$$

where the first sum runs on all the spin of the hypercube and the second one on the lateral spins. The linear dimension of the hypercube can be progressively increased, and one expects to converge to the right result in the limit of large hypercubes. In our opinion, this might be a powerful numerical tool to determine the internal energy and the critical temperature of a spin glass, superior to a direct approach by Monte Carlo simulations.

Let us also mention the major open problem from a theoretical point of view. It is the search of the solution of the Bethe-Peierls equations with replica symmetry breaking, to see whether the unphysical behavior of the internal energy at low temperature disappears as happens in the Parisi solution [12] of the SK model. A first step can be reached by looking for a solution with only one breaking. This can pave the way to the comprehension of the glassy transition in finite dimension.

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